Proofs for the hyperbola

Remember that for a hyperbola we define: (figure 1)

a = distance between the center and a vertex.

c = distance between the center and a focus.

Also, the by the definition of a hyperbola, the absolute value of the difference between the distances from any point on the hyperbola to the foci is a fixed constant. (figure 2) In other words, |d1-d2| has the same value for any point on the hyperbola. Note that the absolute value is used so that we don’t have to be careful about whether d1 or d2 is larger.

Since this constant difference is the same for any point on the hyperbola, let's use the vertex on the right. Looking at figure 3, we can deduce that the constant difference is the distance between the vertices, 2a.

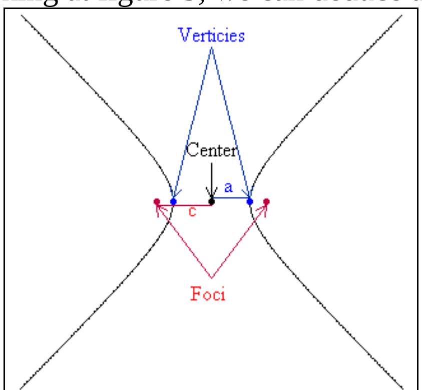


Figure 1

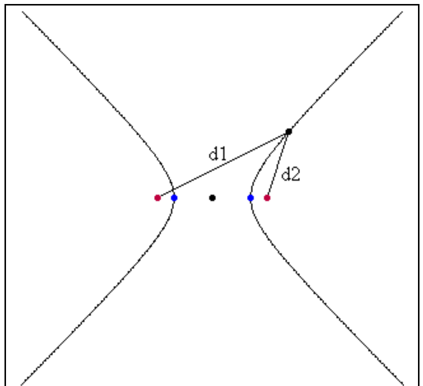


Figure 2

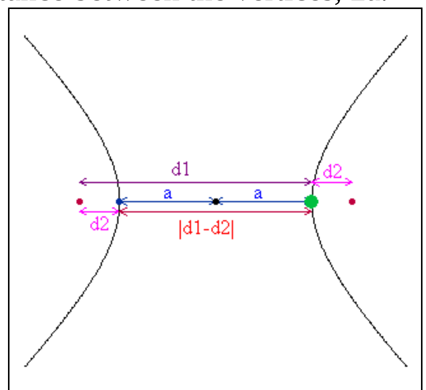


Figure 3

Theorem 1
For any hyperbola the absolute value of the difference between the distances from any point on the hyperbola to the foci is the distance between the vertices. |d1-d2| = 2a.

Let's start with using a hyperbola centered at the origin in the horizontal orientation. Then the graph can be labeled as in figure 4.

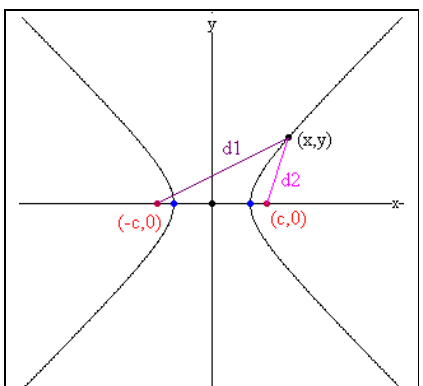


Figure 4

Now using Theorem 1.

$$|d1-d2|=2a$$

$$d1-d2=\pm 2a \quad \text{[The + or − would depend on which side (x, y) is on.]}$$

$$\sqrt{(x+c)^2+(y-0)^2}-\sqrt{(x-c)^2+(y-0)^2}=\pm 2a$$

$$\sqrt{(x+c)^2+y^2}=\pm 2a+\sqrt{(x-c)^2+y^2}$$

$$(x+c)^2+y^2=4a^2\pm 4a\sqrt{(x-c)^2+y^2}+(x-c)^2+y^2$$

$$x^2+2cx+c^2+y^2=4a^2\pm 4a\sqrt{(x-c)^2+y^2}+x^2-2cx+c^2+y^2$$

$$4cx-4a^2=\pm 4a\sqrt{(x-c)^2+y^2}$$

$$cx-a^2=\pm a\sqrt{(x-c)^2+y^2}$$

$$c^2x^2-2a^2cx+a^4=+a^2\left[(x-c)^2+y^2\right]$$

$$c^2x^2-2a^2cx+a^4=a^2\left[x^2-2cx+c^2+y^2\right]$$

$$c^2x^2-2a^2cx+a^4=a^2x^2-2a^2cx+a^2c^2+a^2y^2$$

$$c^2x^2-a^2x^2-a^2c^2+a^4=a^2y^2$$

1 $$x^2\left(c^2-a^2\right)-a^2\left(c^2-a^2\right)=a^2y^2$$

$$\frac{x^2\left(c^2-a^2\right)-a^2\left(c^2-a^2\right)}{a^2}=y^2$$

$$y^2=\frac{x^2\left(c^2-a^2\right)-a^2\left(c^2-a^2\right)}{a^2}$$

$$y=\pm\sqrt{\frac{x^2\left(c^2-a^2\right)-a^2\left(c^2-a^2\right)}{a^2}}$$

$$y=\pm\sqrt{\frac{\left(c^2-a^2\right)}{a^2}x^2-\left(c^2-a^2\right)}$$

From this we want to find out what happens as x get larger.

As $x\to\infty$, the $\frac{\left(c^2-a^2\right)}{a^2}x^2$ will grow to ∞ while the $\left(c^2-a^2\right)$ remains constant, and becomes insignificant. Thus,

$$y\to\pm\sqrt{\frac{\left(c^2-a^2\right)}{a^2}x^2}$$

$$=\pm\frac{\sqrt{\left(c^2-a^2\right)}}{a}x$$

Let's define $b=\sqrt{\left(c^2-a^2\right)}$. Then, we get that

$$\text{As } x\to\infty$$

$$y\to\pm\frac{b}{a}x$$

Therefore, we get the following:

Theorem 2

For a hyperbola centered at the origin in the horizontal orientation, it will have oblique asymptotes, slant asymptotes, of

$$y = \pm\frac{b}{a}x$$

where $b = \sqrt{\left(c^2 - a^2\right)}$.

Note that b and a are related to the rise and run of the O.A. As you will see below, which is which can change.

Now, let's go back to equation [1] above and use that $b = \sqrt{\left(c^2 - a^2\right)}$.

$$b = \sqrt{\left(c^2 - a^2\right)}$$

$$\boxed{2} \quad b^2 = c^2 - a^2$$

$$a^2 + b^2 = c^2$$

$$c^2 = a^2 + b^2$$

$$x^2\left(c^2 - a^2\right) - a^2\left(c^2 - a^2\right) = a^2y^2$$

$$x^2b^2 - a^2b^2 = a^2y^2 \quad \text{using equation } \boxed{2}$$

$$b^2x^2 - a^2y^2 = a^2b^2$$

$$\frac{b^2x^2 - a^2y^2}{a^2b^2} = 1$$

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

Thus,

Theorem 3

For a hyperbola centered at the origin in the horizontal orientation,

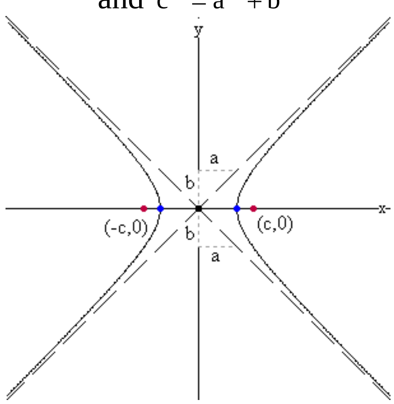
It's equation is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

It will have O.A. of $y = \pm\frac{b}{a}x$

Vertices = $(\pm a, 0)$

Foci = $(\pm c, 0)$

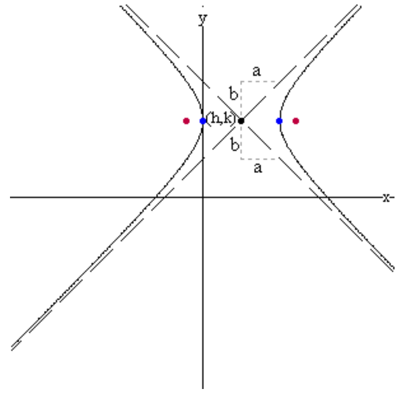
and $c^2 = a^2 + b^2$



Finally, we want to generalize this to any center, (h,k), and for the vertical orientation.

To move the center to a point (h,k) we perform a horizontal shift of h. Thus, we replace x with x-h. Also, we replace y with y-k to perform a vertical shift.

Therefore, getting:

Theorem 4
For a hyperbola in the horizontal orientation, It's equation is $\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$ It will have O.A. of $y=\pm\frac{b}{a}(x-h)+k$ Center = (h, k) Vertices = $(h\pm a, k)$ Foci = $(h\pm c, k)$ and $c^2=a^2+b^2$ 

For the vertical orientation we interchange all x's and y's where appropriate, including the h's and k's and the swapping the rise and run of the O.A.

Theorem 5
For a hyperbola in the vertical orientation, It's equation is $\frac{(y-k)^2}{a^2}-\frac{(x-h)^2}{b^2}=1$ It will have O.A. of $y=\pm\frac{a}{b}(x-h)+k$ Center = (h, k) Vertices = $(h, k\pm a)$ Foci = $(h, k\pm c)$ and $c^2=a^2+b^2$