Proofs for the ellipse

Remember that for an ellipse we define: (figure 1)

a = distance between the center and a vertex.
b = distance from center to endpoint of minor axis.
c = distance between the center and a focus.

Note for an ellipse, a>b.

Also, the by the definition of an ellipse, the sum of the distances from any point on the ellipse to the foci is a fixed constant. (figure 2) In other words, d1+d2 has the same value for any point on the ellipse.

Since this constant sum is the same for any point on the ellipse, let's use the vertex on the right. Looking at figure 3, we can deduce that the constant sum is the distance between the vertices, 2a.

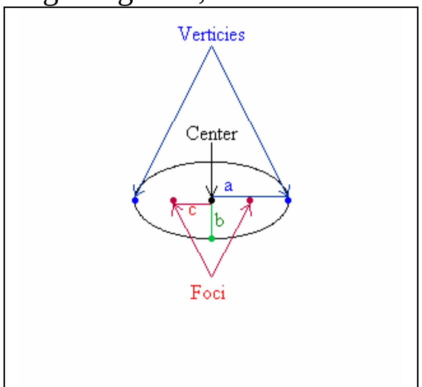


Figure 1

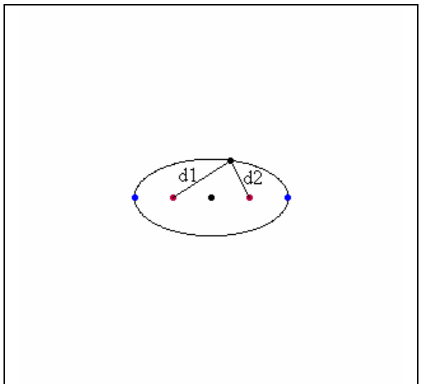


Figure 2

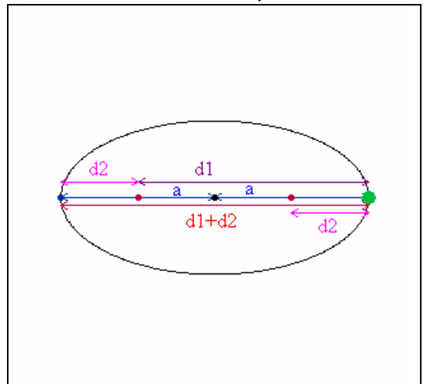


Figure 3

Theorem 1

For any ellipse the sum of the distances from any point on the ellipse to the foci is the distance between the vertices.

$$d1+d2 = 2a.$$

Now, let's use an endpoint of the minor axis. Using figure 4, the fact that in this case d1=d2, and theorem 1, we get that d1=d2=a. Look at figure 5. We now have a right triangle with a hypotenuse of a, and legs of b and c.

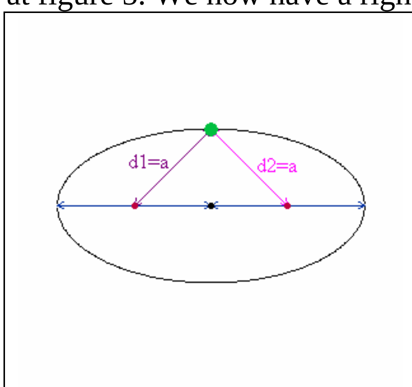


Figure 4

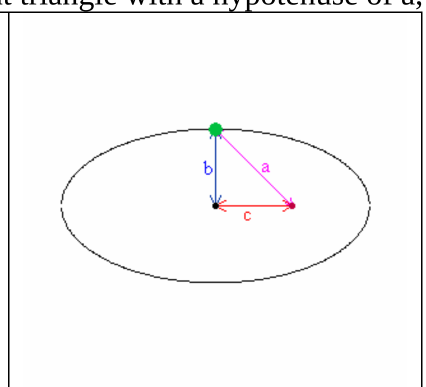


Figure 5

Thus, by the Pythagorean theorem we get:

Theorem 2

For any ellipse,

$$b^2 + c^2 = a^2$$

$$c^2 = a^2 - b^2 \tag{1}$$

Next let's start with using an ellipse centered at the origin in the horizontal orientation. Then, the graph can be labeled as in figure 6.

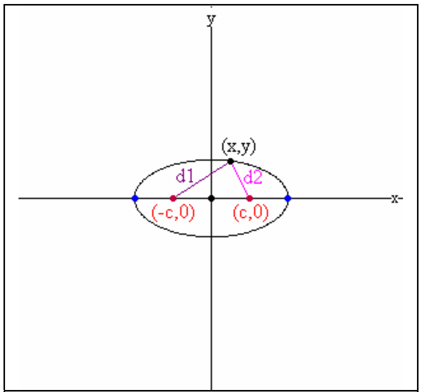


Figure 6

Now using Theorem 1, and equation 1.

$$d1 + d2 = 2a$$

$$\sqrt{(x+c)^2 + (y-0)^2} + \sqrt{(x-c)^2 + (y-0)^2} = 2a$$

$$\sqrt{(x+c)^2 + y^2} = 2a - \sqrt{(x-c)^2 + y^2}$$

$$(x+c)^2 + y^2 = 4a^2 - 4a\sqrt{(x-c)^2 + y^2} + (x-c)^2 + y^2$$

$$x^2 + 2cx + c^2 + y^2 = 4a^2 - 4a\sqrt{(x-c)^2 + y^2} + x^2 - 2cx + c^2 + y^2$$

$$4cx - 4a^2 = -4a\sqrt{(x-c)^2 + y^2}$$

$$cx - a^2 = -a\sqrt{(x-c)^2 + y^2}$$

$$c^2x^2 - 2a^2cx + a^4 = +a^2\left[(x-c)^2 + y^2\right]$$

$$c^2x^2 - 2a^2cx + a^4 = a^2\left[x^2 - 2cx + c^2 + y^2\right]$$

$$c^2x^2 - 2a^2cx + a^4 = a^2x^2 - 2a^2cx + a^2c^2 + a^2y^2$$

$$c^2x^2 - a^2x^2 - a^2c^2 + a^4 = a^2y^2$$

$$\left(a^2 - b^2\right)x^2 - a^2x^2 - a^2\left(a^2 - b^2\right) + a^4 = a^2y^2 \qquad \text{; using } \boxed{1}$$

$$a^2x^2 - b^2x^2 - a^2x^2 - a^4 + a^2b^2 + a^4 = a^2y^2$$

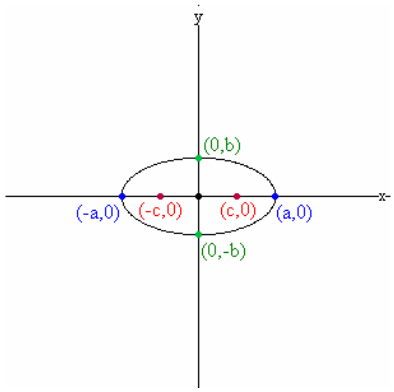
$$-b^2x^2 + a^2b^2 = a^2y^2$$

$$-b^2x^2 - a^2y^2 = -a^2b^2$$

$$b^2x^2 + a^2y^2 = a^2b^2$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

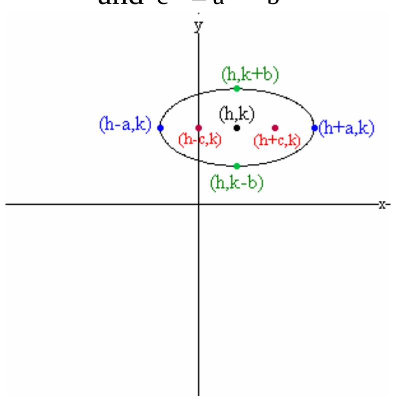
Thus,

Theorem 3
For an ellipse centered at the origin in the horizontal orientation, and a>b, It's equation is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ Center = $(0, 0)$ Vertices = $(\pm a, 0)$ Foci = $(\pm c, 0)$ and $c^2=a^2-b^2$ 

Finally, we want to generalize this to any center, (h,k), and for the vertical orientation.

To move the center to a point (h,k) we perform a horizontal shift of h. Thus, we replace x with x-h. Also, we replace y with y-k to perform a vertical shift.

Therefore, getting:

Theorem 4
For an ellipse in the horizontal orientation, and a>b, It's equation is $\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$ Center = (h, k) Vertices = $(h\pm a, k)$ Foci = $(h\pm c, k)$ and $c^2=a^2-b^2$ 

For the vertical orientation we interchange all x's and y's where appropriate, including the h's and k's.

Theorem 5

For an ellipse in the vertical orientation, and a>b,

It's equation is $\frac{(y-k)^2}{a^2}+\frac{(x-h)^2}{b^2}=1$

Center = (h, k)

Vertices = $(h, k \pm a)$

Foci = $(h, k \pm c)$

and $c^2 = a^2 - b^2$