Proofs of the Logarithmic Properties

Fundamental Properties:

Since $b^0 = 1$, if $b \neq 0$

$$\operatorname{Log}_b(1) = 0$$

$b > 0$ and $b \neq 1$

Because $b^1 = b$

$$\operatorname{Log}_b(b) = 1$$

$b > 0$ and $b \neq 1$

Since $b^x = b^x$

$$\operatorname{Log}_b(b^x) = x$$

$b > 0$ and $b \neq 1$

x is any real

$$\operatorname{Log}_b(x)\cdot(1) = \operatorname{Log}_b(x)$$

$$\operatorname{Log}_b(x)\cdot\operatorname{Log}_b(b) = \operatorname{Log}_b(x)$$

$$\operatorname{Log}_b\left(b^{\operatorname{Log}_b(x)}\right) = \operatorname{Log}_b(x)$$

$$b^{\operatorname{Log}_b(x)} = x$$

Since logs are 1 − 1:

$\operatorname{Log}_b(y) = \operatorname{Log}_b(x)$,

if and only if $y = x$

$$b^{\operatorname{Log}_b(x)} = x$$

$b > 0$ and $b \neq 1$

$x > 0$

Product Rule for Logs

$$\operatorname{Log}_b(x\,y) = \operatorname{Log}_b\left(b^{\operatorname{Log}_b(x)}b^{\operatorname{Log}_b(y)}\right)$$

$$= \operatorname{Log}_b\left(b^{\operatorname{Log}_b(x)+\operatorname{Log}_b(y)}\right)$$

$$= \operatorname{Log}_b(x) + \operatorname{Log}_b(y)$$

$$\operatorname{Log}_b(x\,y) = \operatorname{Log}_b(x) + \operatorname{Log}_b(y)$$

$b > 0$ and $b \neq 1$

x and $y > 0$

Quotient Rule for Logs

$$\operatorname{Log}_b\left(\frac{x}{y}\right) = \operatorname{Log}_b\left(\frac{b^{\operatorname{Log}b(x)}}{b^{\operatorname{Log}b(y)}}\right)$$

$$= \operatorname{Log}_b\left(b^{\operatorname{Log}_b(x)-\operatorname{Log}_b(y)}\right)$$

$$= \operatorname{Log}_b(x) - \operatorname{Log}_b(y)$$

$$\operatorname{Log}_b\left(\frac{x}{y}\right) = \operatorname{Log}_b(x) - \operatorname{Log}_b(y)$$

$b > 0$ and $b \neq 1$

x and $y > 0$

Power Rule for Logs

$$\operatorname{Log}_b(x^n) = \operatorname{Log}_b\left(\left(b^{\operatorname{Log}_b(x)}\right)^n\right)$$

$$= \operatorname{Log}_b\left(b^{n\operatorname{Log}_b(x)}\right)$$

$$= n\operatorname{Log}_b(x)$$

$$\operatorname{Log}_b(x^n) = n\operatorname{Log}_b(x)$$

$b > 0$ and $b \neq 1$

$x > 0$

n is any real

Change of Base Formula

$$\operatorname{Log}_b(x) = \frac{\operatorname{Log}_b(x)\operatorname{Log}_a(b)}{\operatorname{Log}_a(b)}$$

$$= \frac{\operatorname{Log}_a\left(b^{\operatorname{Log}_b(x)}\right)}{\operatorname{Log}_a(b)}$$

$$= \frac{\operatorname{Log}_a(x)}{\operatorname{Log}_a(b)}$$

$$\operatorname{Log}_b(x) = \frac{\operatorname{Log}_a(x)}{\operatorname{Log}_a(b)}$$

$b > 0$ and $b \neq 1$

$a > 0$ and $a \neq 1$

$x > 0$