RATIONALIZING DENOMINATORS AND NUMERATORS

Rewriting a fraction so that there is no radical in the denominator is called rationalizing the denominator. When your fraction has only one term in the denominator you rationalize the denominator by multiplying the fraction by 1 since this will not change the value of the original fraction. You form 1 by putting whatever radical number was in the original denominator over itself.

Let's try some examples.

Rationalize the denominator of $\dfrac{10}{\sqrt{11}}$.

$$\frac{10}{\sqrt{11}} = \frac{10 \cdot \sqrt{11}}{\sqrt{11} \cdot \sqrt{11}} = \frac{10\sqrt{11}}{11}$$

We multiplied the original fraction by $\dfrac{\sqrt{11}}{\sqrt{11}}$ which is 1.

Rationalize the denominator of $\dfrac{\sqrt{54}}{\sqrt{3}}$.

$\dfrac{\sqrt{54}}{\sqrt{3}} = \sqrt{\dfrac{54}{3}}$ Rewrite as a single radical since the numerator and denominator will reduce.

$= \sqrt{18}$ Reduce.

$= \sqrt{9}\sqrt{2}$ Simplify the radical

$= 3\sqrt{2}$ Now there is no denominator left to rationalize.

Rationalize the denominator of $\dfrac{7}{\sqrt{3}}$.

$$\frac{7}{\sqrt{3}} = \frac{7\sqrt{3}}{\sqrt{3}\sqrt{3}} = \frac{7\sqrt{3}}{3}$$

We multiplied the original fraction by $\dfrac{\sqrt{3}}{\sqrt{3}}$ which is 1.

Rationalize the denominator of $\dfrac{18}{\sqrt{12}}$.

$\dfrac{18}{\sqrt{12}} = \dfrac{18}{\sqrt{4}\sqrt{3}}$ Simplify the radical in the denominator.

$= \dfrac{\overset{9}{\cancel{18}}}{\cancel{2}\sqrt{3}} = \dfrac{9}{\sqrt{3}}$ Reduce.

$= \dfrac{9\sqrt{3}}{\sqrt{3}\sqrt{3}}$ Multiply by 1 in the form $\dfrac{\sqrt{3}}{\sqrt{3}}$.

$= \dfrac{9\sqrt{3}}{3} = \dfrac{\overset{3}{\cancel{9}}\sqrt{3}}{\underset{1}{\cancel{3}}}$ Reduce.

$= 3\sqrt{3}$

In all of the above examples we were able to get rid of the radical in the denominator because when we multiply a square root by itself we are left with the radicand only. Remember it takes two in a set to get out from under a square root. To get out from under a cube root it takes a set of 3. Hence if you multiply a cube root by itself that will not help.

For example: $\sqrt[3]{2} \cdot \sqrt[3]{2} = \sqrt[3]{2 \cdot 2} = \sqrt[3]{4}$ We still have the radical since there were not enough 2's to bring it out.

Now consider $\sqrt[3]{2} \cdot \sqrt[3]{2^2} = \cdot \sqrt[3]{2^3} = 2$ We started with one 2 then multiplied by a radical that had two more 2's for a total of three. With a cube root it takes three in a set to get out from under the radical.

Rationalize the denominator of $\dfrac{2}{\sqrt[3]{25}}$.

$\dfrac{2}{\sqrt[3]{25}} = \dfrac{2}{\sqrt[3]{5^2}}$ Rewrite the radicand in factored form as a power of 5.

$= \dfrac{2}{\sqrt[3]{5^2}} \cdot \dfrac{\sqrt[3]{5}}{\sqrt[3]{5}}$ Multiply by 1 in the form $\dfrac{\sqrt[3]{5}}{\sqrt[3]{5}}$. Note we had two 5's originally and just needed one more.

$= \dfrac{2\sqrt[3]{5}}{\sqrt[3]{5^3}}$ Multiply

$= \dfrac{2\sqrt[3]{5}}{5}$ We have three 5's under the radical, hence it comes out.

Rationalize the denominator of $\dfrac{4}{\sqrt[3]{3}}$.

$\dfrac{4}{\sqrt[3]{3}} = \dfrac{4}{\sqrt[3]{3^1}}$ Notice there is only one factor of 3 under the radical. That means we need two more.

$= \dfrac{4}{\sqrt[3]{3}} \cdot \dfrac{\sqrt[3]{3^2}}{\sqrt[3]{3^2}}$ Multiply by 1 in the form $\dfrac{\sqrt[3]{3^2}}{\sqrt[3]{3^2}}$.

$= \dfrac{4\sqrt[3]{3^2}}{\sqrt[3]{3^3}}$ Multiply

$= \dfrac{4\sqrt[3]{9}}{3}$ We have three 3's under the radical, hence it comes out.

Now let's look at how to rationalize when our denominator has two terms instead of just one.

Note: $(2+\sqrt{3})(2-\sqrt{3}) = 4 - 3 = 1$

When we FOIL conjugates the outer and inner products subtract out and all we are left with is the first and last product. Note that all radicals are gone. This is what we will make use of to rationalize when the denominator has two terms. We will multiply the numerator and denominator by the conjugate of the denominator. Remember to form a conjugate you only switch the middle sign.

Rationalize the denominator of $\dfrac{7}{3-\sqrt{5}}$.

$\dfrac{7}{3-\sqrt{5}} = \dfrac{7}{3-\sqrt{5}} \cdot \dfrac{3+\sqrt{5}}{3+\sqrt{5}}$ Multiply numerator and denominator by conjugate of the denominator.

$= \dfrac{7(3+\sqrt{5})}{(3-\sqrt{5})(3+\sqrt{5})}$

$= \dfrac{7(3+\sqrt{5})}{9-5}$ $(3-\sqrt{5})(3+\sqrt{5}) = 9 - 5$ since inner and outer products subtract out.

$= \dfrac{7(3+\sqrt{5})}{4}$ This won't reduce so go ahead and distribute in the numerator.

$= \dfrac{21 + 7\sqrt{5})}{4}$

Rationalize the denominator of $\dfrac{10}{4-\sqrt{11}}$.

$$\frac{10}{4-\sqrt{11}} = \frac{10}{4-\sqrt{11}} \cdot \frac{4+\sqrt{11}}{4+\sqrt{11}}$$

Multiply numerator and denominator by conjugate of the denominator.

$$= \frac{10\left(4+\sqrt{11}\right)}{\left(4-\sqrt{11}\right)\left(4+\sqrt{11}\right)}$$

$$= \frac{10\left(4+\sqrt{11}\right)}{16-11}$$

$(4-\sqrt{11})(4+\sqrt{11}) = 16-11.$

$$= \frac{10\left(4+\sqrt{11}\right)}{5}$$

$$= \frac{\overset{2}{\cancel{10}}\left(4+\sqrt{11}\right)}{\cancel{5}}$$

Reduce.

$$= 2\left(4+\sqrt{11}\right) \text{ or } 8+2\sqrt{11}$$

Rationalize the denominator of $\dfrac{\sqrt{3}}{2+\sqrt{3}}$.

$$\frac{\sqrt{3}}{2+\sqrt{3}} = \frac{\sqrt{3}}{2+\sqrt{3}} \cdot \frac{2-\sqrt{3}}{2-\sqrt{3}}$$

Multiply numerator and denominator by conjugate of the denominator.

$$= \frac{\sqrt{3}\left(2-\sqrt{3}\right)}{\left(2+\sqrt{3}\right)\left(2-\sqrt{3}\right)}$$

$$= \frac{2\sqrt{3}-\sqrt{3}\sqrt{3}}{4-3}$$

Distribute in numerator and multiply conjugates in denominator.

$$= \frac{2\sqrt{3}-3}{1}$$

Division by 1 does not change the numerator.

$$= 2\sqrt{3}-3$$

Rationalize the denominator of $\dfrac{6}{\sqrt{x}+\sqrt{3}}$.

$$\frac{6}{\sqrt{x}+\sqrt{3}} = \frac{6}{\sqrt{x}+\sqrt{3}} \cdot \frac{\sqrt{x}-\sqrt{3}}{\sqrt{x}-\sqrt{3}}$$

Multiply numerator and denominator by conjugate of the denominator.

$$= \frac{6\left(\sqrt{x}-\sqrt{3}\right)}{\left(\sqrt{x}+\sqrt{3}\right)\left(\sqrt{x}-\sqrt{3}\right)}$$

$$= \frac{6\left(\sqrt{x}-\sqrt{3}\right)}{x-3}$$

Note: $\left(\sqrt{x}+\sqrt{3}\right)\left(\sqrt{x}-\sqrt{3}\right) = x-3$

$$= \frac{6\sqrt{x}-6\sqrt{3}}{x-3}$$

Rationalize the denominator of $\dfrac{\sqrt{5}+\sqrt{3}}{\sqrt{5}-\sqrt{3}}$.

$$\frac{\sqrt{5}+\sqrt{3}}{\sqrt{5}-\sqrt{3}} = \frac{\sqrt{5}+\sqrt{3}}{\sqrt{5}-\sqrt{3}} \cdot \frac{\sqrt{5}+\sqrt{3}}{\sqrt{5}+\sqrt{3}}$$

Note the conjugate of the denominator is $\sqrt{5}+\sqrt{3}$.

$$= \frac{\left(\sqrt{5}+\sqrt{3}\right)\left(\sqrt{5}+\sqrt{3}\right)}{\left(\sqrt{5}-\sqrt{3}\right)\left(\sqrt{5}+\sqrt{3}\right)}$$

$$= \frac{\sqrt{5}\sqrt{5}+\sqrt{5}\sqrt{3}+\sqrt{3}\sqrt{5}+\sqrt{3}\sqrt{3}}{5-3}$$

FOIL

$$= \frac{5+\sqrt{15}+\sqrt{15}+3}{5-3}$$

$$= \frac{8+2\sqrt{15}}{2}$$

Combine like terms.

$$= \frac{\cancel{2}\left(4+\sqrt{15}\right)}{\cancel{2}}$$

Factor and reduce.

$$= 4+\sqrt{15}$$

Sometimes it is necessary to rationalize a numerator. That is done in a similar fashion. Instead of multiplying by the radical in the denominator you multiply by the radical in the numerator. Or if there are two terms in the numerator you multiply by the conjugate of the numerator.

Rationalize the numerator of $\dfrac{\sqrt{7}}{3}$.

$$\frac{\sqrt{7}}{3} = \frac{\sqrt{7}}{3} \cdot \frac{\sqrt{7}}{\sqrt{7}} = \frac{\sqrt{7}\sqrt{7}}{3\sqrt{7}} = \frac{7}{3\sqrt{7}}$$

We multiplied the original fraction by $\dfrac{\sqrt{7}}{\sqrt{7}}$ which is 1.

We no longer have a radical in the numerator.

Rationalize the numerator of $\dfrac{7-\sqrt{5}}{3}$

$$\frac{7-\sqrt{5}}{3} = \frac{7-\sqrt{5}}{3} \cdot \frac{7+\sqrt{5}}{7+\sqrt{5}}$$

Multiply numerator and denominator by conjugate of the numerator.

$$= \frac{(7-\sqrt{5})(7+\sqrt{5})}{3(7+\sqrt{5})}$$

$$= \frac{49-5}{3(7+\sqrt{5})}$$

$(7-\sqrt{5})(7+\sqrt{5}) = 49-5$ since inner and outer products subtract out.

$$= \frac{44}{3(7+\sqrt{5})}$$

This won't reduce so go ahead and distribute in the denominator.

$$= \frac{44}{21+3\sqrt{5})}$$