COMPLETING THE SQUARE

In the past we have solved some quadratic equations. The method that we were able to use was the factoring method. For example, solve $x^2 + 5x + 6 = 0$.

$$x^2 + 5x + 6 = 0$$
$$(x+2)(x+3) = 0$$
$$x + 2 = 0 \qquad x + 3 = 0$$
$$x = -2 \qquad x = -3$$

Factoring is a good method to use, but it is necessary for the polynomial to factor in order to be able to use it. We know that some polynomials are prime, which means they will not factor. We need to be able to solve even if it will not factor. Completing the square method can be used to solve a polynomial equation whether it factors or not.

Let's recall what it means for a trinomial to be a **perfect square trinomial**. It means that the factored form of that trinomial is a binomial squared. In other words the factors are identical. For example,
$x^2 + 6x + 9 = (x+3)(x+3) = (x+3)^2$

Hence $x^2 + 6x + 9$ is a perfect square trinomial since its factors $x + 3$ and $x + 3$ are identical.

$x^2 + 5x + 6 = (x+2)(x+3)$

$x^2 + 5x + 6$ is **not** a perfect square trinomial because its two factors $x + 2$ and $x + 3$ are not alike, they are different.

The method of completing the square relies on us being able to create perfect square trinomials. For example if we have the terms $x^2 + 6x$ we could turn this into a perfect square trinomial by adding 9. In that $x^2 + 6x + 9$ is a perfect square trinomial like we saw above. The question is, how do we know what number will make it a perfect square trinomial. This is easy because there is a formula. Take one-half of the x-coefficient and square it. In $x^2 + 6x$, 6 is the coefficient of x. Take one-half of 6 which is 3 and then square it. 3^2 is 9. This is the number that made $x^2 + 6x$ a perfect square trinomial.

Let's try another one.
$x^2 + 8x + ?$ is a perfect square trinomial? Use our formula. $\frac{1}{2}(8) = 4,$ then square it. $4^2 = 16$. The number we are looking for is 16. Check it. $x^2 + 8x + 16 = (x+4)^2$. It factors to a binomial squared, so it is a perfect square trinomial.

One more: $x^2 + 12x + ?$
$\left[\frac{1}{2}(12)\right]^2 = (6)^2 = 36$ The number we need is 36.
Check it: $x^2 + 12x + 36 = (x+6)^2$

Now that we know how to form a perfect square trinomial let's look at this method of solving a quadratic equation called completing the square.

Steps for Completing the Square

1. Get variable terms on one side of the equation and the constant on the other side of the equation.
2. Make sure the coefficient of x^2 is 1. If it is not then divide both sides of the equation by whatever the coefficient is.

These two steps must be done first before the remaining steps. If the coefficient of x^2 is not 1, then the formula we used above to find the correct number to turn it into a perfect square will not work.

3. Turn the variable side of the equation into a perfect square trinomial. Use the formula $\left[\frac{1}{2}(x\text{-coefficient})\right]^2$. Add the number you get from the formula to both sides of the equation. Remember you must keep the equation balanced by adding the same thing to both sides.
4. Factor the variable side. Since it is a perfect square trinomial it will factor into a binomial squared. In fact you will already know the binomial from the work you just did. It will be in the form $\left[x+\frac{1}{2}(x\text{-coefficient})\right]^2$. You also need to simplify the constant side.
5. To get rid of the square, take the square root of both sides of the equation.
6. Isolate x. This is your solution set.

Let's look at some examples.

Solve:			
	$x^2+8x+6=0$		Original equation
	$x^2+8x=-6$		Get variables on one side, constant on the other side.
	$+16 \quad +16$	$\left[\frac{1}{2}(8)\right]^2=(4)^2=16$	Use formula to find the correct number to add.
	$x^2+8x+16=10$		
	$(x+4)^2=10$		Factor the variable side.
	$x+4=\pm\sqrt{10}$		Take the square root of both sides.
	$x=-4\pm\sqrt{10}$		Isolate x. This is the solution set.

Note in this example that the x-coefficient is 8. That is why we took one-half of 8 and then squared it. This gave us 16 which we then added to both sides of the equation. When we took one-half of 8 we got **4**. Notice in the factored form of the perfect square trinomial it is $x+\mathbf{4}$ as the base.

Solve:			
	$x^2-4x-3=0$		Original equation
	$x^2-4x=3$		Get variables on one side, constant on the other side.
	$+4 \quad +4$	$\left[\frac{1}{2}(-4)\right]^2=(-2)^2=4$	Use formula to find the correct number to add.
	$x^2-4x+4=7$		
	$(x-2)^2=7$		Factor the variable side.
	$x-2=\pm\sqrt{7}$		Take the square root of both sides.
	$x=2\pm\sqrt{7}$		Isolate x. This is the solution set.

Note in this example that the x-coefficient is -4. That is why we took one-half of -4 and then squared it. This gave us 4 which we then added to both sides of the equation. When we took one-half of -4 we got $-\mathbf{2}$. Notice in the factored form of the perfect square trinomial it is $x+-\mathbf{2}$ or $x-2$ as the base.

Solve:

$2z^2 - 12z + 4 = 0$	Original equation
$2z^2 - 12z = -4$	Get variables on one side, constant on the other side.
$z^2 - 6z = -2$	Divide both sides by 2 to make the z^2 coefficient 1.
$+9 \quad +9 \qquad \left[\frac{1}{2}(-6)\right]^2 = (-3)^2 = 9$	Use formula to find the correct number to add.
$z^2 - 6z + 9 = 7$	
$(z-3)^2 = 7$	Factor the variable side.
$z - 3 = \pm\sqrt{7}$	Take the square root of both sides.
$z = 3 \pm \sqrt{7}$	Isolate z. This is the solution set.

Solve:

$2x^2 + 3 = 6x$	Original equation
$2x^2 - 6x + 3 = 0$	Get variables on one side.
$2x^2 - 6x = -3$	Get constant on the other side.
$x^2 - 3x = -\frac{3}{2}$	Divide both sides by 2 to make the x^2 coefficient 1.
$+\frac{9}{4} \quad +\frac{9}{4}$	$\left[\frac{1}{2}(-3)\right]^2 = (-\frac{3}{2})^2 = \frac{9}{4}$ Add this number to both sides.
$x^2 - 3x + \frac{9}{4} = -\frac{3}{2} + \frac{9}{4}$	
$\left(x - \frac{3}{2}\right)^2 = \frac{3}{4}$	Factor the variable side. Simplify constant side.
$x - \frac{3}{2} = \pm\sqrt{\frac{3}{4}}$	Take the square root of both sides.
$x - \frac{3}{2} = \pm\frac{\sqrt{3}}{2}$	Simplify radical.
$x = \frac{3}{2} \pm \frac{\sqrt{3}}{2} = \frac{3 \pm \sqrt{3}}{2}$	Isolate x. This is the solution set.

Solve:

$4z^2 + 20z + 19 = 0$	Original equation
$4z^2 + 20z = -19$	Get variables on one side, constant on the other side.
$z^2 + 5z = -\frac{19}{4}$	Divide both sides by 4 to make the z^2 coefficient 1.
$+\frac{25}{4} \quad +\frac{25}{4}$	$\left[\frac{1}{2}(5)\right]^2 = (\frac{5}{2})^2 = \frac{25}{4}$ Add this number to both sides.
$z^2 + 5z + \frac{25}{4} = \frac{-19}{4} + \frac{25}{4}$	
$\left(z + \frac{5}{2}\right)^2 = \frac{6}{4}$	Factor the variable side. Simplify constant side.
$z + \frac{5}{2} = \pm\sqrt{\frac{6}{4}}$	Take the square root of both sides.
$z + \frac{5}{2} = \pm\frac{\sqrt{6}}{2}$	Simplify radical.
$z = -\frac{5}{2} \pm \frac{\sqrt{6}}{2}$	Isolate z. This is the solution set.
$z = \frac{-5 \pm \sqrt{6}}{2}$	It can also be written this way with the fractions combined.